Local properties of solutions to non-autonomous parabolic PDEs with state-dependent delays [1]

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Abstract. A wide class of non-autonomous nonlinear parabolic partial differential equations with delay is studied. We allow in our investigations different types of delays such as constant, time-dependent, state-dependent (both discrete and distributed) to be presented simultaneously. The main difficulties arise due to the presence of discrete state-dependent delays since the nonlinear delay term is not Lipschitz on the space of continuous functions. We find conditions for the local existence, uniqueness and study the invariance principle.

1 Introduction

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We consider non-autonomous parabolic partial differential equations (PDEs) with delay. Studying of this type of equations is based on 1) the well-developed theories of the delayed ordinary differential equations (ODEs) [10, 11, 6] and 2) PDEs without delays [8, 9, 14, 15]. Under certain assumptions both types of equations describe a kind of dynamical systems that are infinite-dimensional, see [1, 29, 5] and references therein; see also [30, 7, 18, 3, 4] and the monograph [34] that are close to our work.

In evolution systems arising in applications the presented delays are frequently state-dependent (SDDs). The theory of such equations, especially the ODEs, is rapidly developing and many deep results have been obtained up to now (see e.g. [20, 31, 32, 17] and also the survey paper [12] for details and references). The PDEs with state-dependent delays were first studied in [21, 13, 22]. An alternative approach to the PDEs with discrete SDDs is proposed in [23]. Approaches to equations with discrete and distributed SDDs are different. Even in the case of ODEs, the discrete SDD essentially complicates the study since, in general, the corresponding *nonlinearity* is *not* locally Lipschitz continuous on open subsets of the space of continuous functions, and familiar results on existence, uniqueness, and dependence of solutions on initial data and parameters from, say [11, 6] fail (see [33] for an example of the non-uniqueness and [12] for more details). It is important to mention that due to the discrete SDDs such equations are inherently *nonlinear*. In this work, in contrast to previous investigations, we consider a model where two different types of SDDs (discrete and distributed) are presented simultaneously (by Stieltjes integral). Moreover, all the assumptions on the delay (see (A1)-(A5) below) allow the dynamics when along a solution the number and values of discrete SDDs may change, the whole discrete and/or distributed delays may vanish, disappear and appear again. This property makes it possible to study "flexible" models where some subsets of the phase space are described by equations with purely discrete SDDs, and others by equations with purely distributed SDDs, and there are subsets which need the general (combined) type

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of the delay. A solution could be in different subsets at different time moments. This property particularly means that not only the values of the delays are state-dependent, but the *type* of the delay is *state-dependent* as well.

The first goal of the present paper is to study the basic properties of solutions - the existence and uniqueness as well as to extend the fundamental invariance principle to the case of PDEs with discrete SDDs. The second goal is to attract attention of researchers from such fields as, for example, mathematical biology and physics to this wide class of delay equations and emphasize that the crucial assumption on the delay (see (A5) below) is an "inner property" of the delay which could be successfully used in a wide range of other delay systems. We hope that our results provide a basement for further study of qualitative (asymptotic) properties of solutions.

The existence and uniqueness results for a particular case of autonomous systems were announced in [25]. For a survey of the existing literature on the invariance principle see [26]. In the present paper the emphasis is on the delayed term, not on the partial differential operator. To the best of our knowledge the invariance principle for PDEs with SDDs has not been studied before.

2 Formulation of the model and examples

Let X be a Banach space with the norm $||\cdot||$, let $r>0$ be a constant. Denote by $C\equiv C([-r,0];X)$ the space of continuous functions $\varphi:[-r,0]\to X$ with the supremum norm $||\cdot||_C$. As usually for delay equations [10, 11], for any real $a\le b, t\in[a,b]$ and any continuous function $u:[a-r,b]\to X$, we denote by u_t the element of C defined by the formula $u_t=u_t(\theta)\equiv u(t+\theta)$ for $\theta\in[-r,0]$. Consider an infinitesimal generator A of a (compact) C_0 semigroup $\{e^{-At}\}_{t\ge 0}\equiv\{T(t)\}_{t\ge 0}$ on X satisfying $||T(t)||\le e^{\omega t}$ for all $t\ge 0$, where $\omega\in\mathbb{R}$ is a fixed constant.

We are interested in the following non-autonomous parabolic partial differential equation with state-dependent delays (SDD)

$$\frac{du(t)}{dt}+Au(t)=B\big(t,u_t\big),\quad t\ge a \tag{1}$$

with the initial condition

$$u_a=u|_{[a-r,a]}=\varphi\in C\equiv C([-r,0];X). \tag{2}$$

The delay term $B:\mathbb{R}\times C\to X$ has the form

$$B\big(t,\psi\big)\equiv G\big(t,\psi(0),F(t,\psi)\big), \tag{3}$$

where $G:\mathbb{R}\times X\times X\to X$ is a continuous mapping and the delay functional $F:\mathbb{R}\times C\to X$ is presented by a Stieltjes integral (simultaneously includes discrete and distributed SDDs)

$$F(t,\psi)\equiv\int_{-r}^{0}p\left(t,\psi(\theta)\right)\cdot dg(\theta,t,\psi),\quad p:\mathbb{R}\times X\to X. \tag{4}$$

Assumptions on g are formulated below (see (A1)-(A5)).

The class of equations described by (1),(3),(4) is very wide and includes many equations which were intensively studied during past decades. Below we mention just two examples and refer the reader to [34, 21] for more references and discussion.

Example 1. Let $X = L^2(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. Operator A is a densely-defined self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and compact resolvent, which means that $A : D(A) \to L^2(\Omega)$ generates an analytic semigroup. If we choose $G(t, u, v) = v - du$ ($d \geq 0$ is a constant), then $B\big(t, u_t\big) = F(t, u_t) - du$ and equation (1) reads as

$$\frac{\partial}{\partial t}u(t,x) + Au(t,x) + du(t,x) = \big(F(u_t)\big)(x), \tag{5}$$

with, for example,

$$\big(F(\psi)\big)(x) \equiv \int_{-r}^{0} \left\{ \int_{\Omega} p\left(\psi(\theta,y)\right) f(x-y)dy \right\} \cdot dg(\theta,\psi), \quad x \in \Omega, \tag{6}$$

where $f : \Omega - \Omega \to \mathbb{R}$ is a bounded measurable function, $p : \mathbb{R} \to \mathbb{R}$. This non-local *autonomous* equation is studied in [25]. It is clear that the integral delay term given by (6) includes the cases:
a) purely discrete SDDs: $\big(F(\psi)\big)(x) = \sum_k \int_\Omega p\left(\psi(-\eta_k(\psi), y)\right) f(x-y)\, dy$;
b) purely distributed SDD: $\big(F(\psi)\big)(x) = \int_{-r}^{0} \left\{\int_\Omega p\left(\psi(\theta,y)\right) f(x-y)dy\right\} \cdot \xi(\theta,\psi)\, d\theta$. These cases have been studied in [21, 22, 23].

Similarly, one may consider *local* delay terms (discrete and/or distributed SDD)

$$\big(F(\psi)\big)(x) \equiv \int_{-r}^{0} p\left(\psi(\theta,x)\right) \cdot dg(\theta,\psi), \quad x \in \Omega. \tag{7}$$

The above type of equations includes the diffusive Nicholson's blowflies equation (see e.g. [28]) with state-dependent delays, i.e. equation (5) where $-A$ is the Laplace operator with Dirichlet or Neumann boundary conditions, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary, the nonlinear (birth) function p is given by $p(w) = p_1 \cdot we^{-w}$, $p_1 \in \mathbb{R}$.

Example 2 (`reaction-diffusion system with delay`). Suppose $\Omega \subset \mathbb{R}^N$ is a bounded region with a smooth boundary $\partial\Omega$, ∂_n is the outward normal derivative on $\partial\Omega$, $\triangle$ is the Laplacian operator on Ω. Consider the system

$$\begin{cases} \frac{\partial u^i}{\partial t}(x,t) = d_i \triangle u^i(x,t) + G^i(t, F^i(t, u_t^1(x,\cdot), \ldots, u_t^m(x,\cdot))), & t > a,\ x \in \Omega, \\ \alpha^i(x)u^i(x,t) + \partial_n u^i(x,t) = 0, & t > a,\ x \in \partial\Omega, \\ u^i(x, a+\theta) = \varphi^i(x,\theta), & \theta \in [-r, 0],\ x \in \Omega, \end{cases} \tag{8}$$

where $i = 1, \ldots, m$. In (8), $d_i \geq 0$ and $d_i = 0$ we agree that no boundary condition applies to u^i, $\alpha^i \in C^{1+\alpha}(\partial\Omega), \alpha \in (0,1)$. Functions $G^i : \mathbb{R}^2 \to \mathbb{R}$ are locally Lipschitz and the delay functionals $F^i : \mathbb{R} \times C([-r,0]; \mathbb{R}^m) \to \mathbb{R}$ are presented by Stieltjes integral (simultaneously includes discrete and distributed SDDs) similar to (4). The system (8) could be presented in the form (1)-(3) as follows. We set $X \equiv C(\bar{\Omega}; \mathbb{R}^m)$ and (see e.g. [19, p.5], [18] and references therein) let A_i^0 be the operator defined by $A_i^0 y_i = d_i \Delta y_i$

(or $A_i^0 y_i = 0$ if $d_i = 0$) on the domain $D(A_i^0) \equiv \{y_i \in C^1(\bar\Omega) \cap C^2(\Omega);\ \alpha^i y^i + \partial_n y^i = 0 \text{ on } \partial\Omega\}$ (or $D(A_i^0) \equiv C(\bar\Omega)$ if $d_i = 0$)). The operator A_i is the closure of A_i^0 on $C(\bar\Omega)$ and $A \equiv (A_i)_{i=1}^m$. We denote $\{T_i(t)\}_{t\geq 0}$ the C_0-semigroup on $C(\bar\Omega)$ generated by A_i and $T \equiv (T_i)_{i=1}^m$. It is well-known [18] that T is a C_0-semigroup on $X = C(\bar\Omega; \mathbb{R}^m)$ that is analytic (and compact if all $d_i > 0$) and A is its generator. The system (8) was studied (without state-dependent delays), for example, in [18].

As an application one can consider the n-species Lotka-Volterra model of competition with diffusion and delays given by

$$\begin{cases} \frac{\partial u^i}{\partial t}(x,t) = d_i \triangle u^i(x,t) + b_i u^i(t,x) \left[1 - \sum_{j=1}^n c_{ij} \int_{-r}^0 u^j(x, t+\theta))\, dg_{ij}(\theta, u_t)\right], & x \in \Omega, \\ \partial_n u^i(x,t) = 0, & x \in \partial\Omega, \\ u^i(x, a+\theta) = \varphi^i(x,\theta), \quad \theta \in [-r, 0], & \end{cases} \tag{9}$$

where b_i, c_{ij} are positive constants and g_{ij} are nondecreasing with respect to the first coordinate and $g_{ij}(0,\cdot) - g_{ij}(-r,\cdot) = 1$. Many interesting properties of this system (autonomous and without state-dependent delays) were discussed in [19] (see also references therein).

The approach developed in the present article is applicable to more general classes of equations of the form (1) with the nonlinearity B, for example, as follows (c.f. (3))

$$B\big(t, u_t\big) \equiv G\big(t, F^1(t,u_t), \ldots, F^k(t,u_t)\big),$$

with F^i be as in (4). We formulate our results for B given by (3) for the simplicity of presentation and motivated by (9).

3 Local existence and uniqueness

The following assumptions on the *time*- and *state*-dependent delay are generalizations to the non-autonomous case of the ones proposed in [25].

(A1) *For any $(t,\varphi) \in \mathbb{R} \times C$, the function $[-r,0] \ni g(\cdot,t,\varphi) \to \mathbb{R}$ is of bounded variation on $[-r,0]$. The variation $V_{-r}^0 g$ of g is* `uniformly bounded` *i.e.*
$\exists M_{Vg} > 0 : \forall (t,\varphi) \in \mathbb{R} \times C \quad \Rightarrow \quad V_{-r}^0 g(\cdot,t,\varphi) \leq M_{Vg}$.

It is well-known that any Lebesgue-Stieltjes measure (associated with g) may be split into a sum of three measures: discrete, absolutely continuous and singular ones. We will denote the corresponding splitting of g as follows

$$g(\theta,t,\varphi) = g_d(\theta,t,\varphi) + g_{ac}(\theta,t,\varphi) + g_s(\theta,t,\varphi) = g_d(\theta,t,\varphi) + g_c(\theta,t,\varphi), \tag{10}$$

where $g_d(\theta,t,\varphi)$ is a step-function, $g_{ac}(\theta,t,\varphi)$ is absolutely continuous and $g_s(\theta,t,\varphi)$ is singular continuous as functions of their first coordinates (see [16] for more details) and we denote the continuous part by $g_c \equiv g_{ac} + g_s$.

Our next assumptions are

(A2) *For any $\theta \in [-r,0]$, the function g_c is continuous with respect to its second and third coordinates i.e. $\forall \theta \in [-r,0], \quad \forall (t,\varphi), (t^n,\varphi^n) \in \mathbb{R} \times C : (t^n,\varphi^n) \to (t,\varphi)$ in $\mathbb{R} \times C$ $(n \to +\infty) \Rightarrow g_c(\theta,t^n,\varphi^n) \to g_c(\theta,t,\varphi)$.*

(A3) *The step-function $g_d(\theta, t, \varphi)$ is continuous with respect to (t, φ) in the sense that discontinuities of $g_d(\theta, t, \varphi)$ at points $\{\theta_k\} \subset [-r, 0]$ satisfy the property: there are* `continuous` *functions $\eta_k : \mathbb{R} \times C \to [0, r]$ and $h_k : \mathbb{R} \times C \to \mathbb{R}$ such that $\theta_k = -\eta_k(t, \varphi)$ and $h_k(t, \varphi)$ is the jump of g_d at point $\theta_k = -\eta_k(t, \varphi)$ i.e $h_k(t, \varphi) \equiv g_d(\theta_k + 0, t, \varphi) - g_d(\theta_k - 0, t, \varphi)$.*

Taking into account that g_d may, in general, have infinite (countable) number of points of discontinuity $\{\theta_k\}$, we assume that the series $\sum_k h_k(t, \varphi)$ converges `absolutely` *and* `uniformly on any bounded subsets of` $\mathbb{R} \times C$.

Following notations of (10), we conclude that (A3) means that for any $(t, \chi) \in \mathbb{R} \times C$ one has $\Phi_d(t, \chi) \equiv \int_{-r}^{0} \chi(\theta)\, dg_d(\theta, t, \varphi) = \sum_k \chi(\theta_k) \cdot h_k(t, \varphi) = \sum_k \chi(-\eta_k(t, \varphi)) \cdot h_k(t, \varphi)$. Here all η_k and h_k are `continuous` functions.

The first result is (c.f. [25, lemma 1])

Theorem 1. *Assume $G : \mathbb{R} \times X \times X \to X$ is a continuous mapping and $p : \mathbb{R} \times X \to X$ (see (4)) is Lipschitz ($||p(t, u) - p(s, v)|| \le L_p(|s - t| + ||u - v||)$, satisfying $||p(s, u)|| \le C_1||u|| + C_2, \forall (s, u) \in \mathbb{R} \times X$ with $C_i \ge 0$. Under assumptions (A1)- (A3), the nonlinear mapping $B : \mathbb{R} \times C \to X$, defined by (3), is continuous.*

Remark. *It is important that nonlinear map B is not Lipschitz in the presence of discrete SDDs. The last means that discrete delays may be present, but be constant or time-dependent only (i.e. $g_d(\theta, t, \varphi) = \widehat{g}_d(\theta, t)$).*

Proof of theorem 1. Since a composition of continuous mappings is continuous, it is enough (see (3)) to show the continuity of F defined by (4).

We first split our g in continuous and discontinuous parts $g_c \equiv g_{ac} + g_s$ and g_d, respectively (see (10)). This splitting gives the corresponding splitting $F = F_c + F_d$, where F_c corresponds to the continuous part $g_c \equiv g_{ac} + g_s$.

Case 1. Let us first consider the *part* F_c. We write

$$F_c(t^1, \varphi) - F_c(t^2, \psi) = I_1 + I_2, \tag{11}$$

where we denote

$$I_1 = I_1(\varphi, \psi) \equiv \int_{-r}^{0} \left[p(t^1, \varphi(\theta)) - p(t^2, \psi(\theta)) \right]\, dg_c(\theta, t^1, \varphi), \tag{12}$$

$$I_2 = I_2(\varphi, \psi) \equiv \int_{-r}^{0} p(t^2, \psi(\theta))\, d\,[g_c(\theta, t^1, \varphi) - g_c(\theta, t^2, \psi)]. \tag{13}$$

Using the Lipschitz property of p and (A1), one can check that

$$||I_1|| \le L_p(|t^1 - t^2| + ||\varphi - \psi||_C) \cdot M_{Vg}. \tag{14}$$

This shows that $||I_1|| \to 0$ when $|t^1 - t^2| + ||\varphi - \psi||_C \to 0$. To show that $||I_2|| \to 0$ (when $t^1 \to t^2$ and $\varphi \to \psi$ in C) we use assumptions (A1) and (A2) to apply the first Helly's theorem [16, page 359].

Case 2. Now we prove the continuity of F_d (*discrete delays*). Let us fix any $\varphi \in C, t \in \mathbb{R}$ and consider any sequences $\{\varphi^n\} \subset C$ and $\{t^n\} \subset \mathbb{R}$ such that $||\varphi^n - \varphi||_C \to 0$ and $t^n \to t$ when $n \to \infty$. Our goal is to prove that $||F_d(t^n \varphi^n) - F_d(t, \varphi)|| \to 0$.

Following the notations of (A3) we write

$$F_d(t,\varphi) = \sum_k p(t,\varphi(-\eta_k(t,\varphi))) \cdot h_k(t,\varphi)$$

and remind that it could be a series or a finite sum. We split as follows

$$F_d(t^n,\varphi^n) - F_d(t,\varphi) \equiv K_1^n + K_2^n + K_3^n \in X, \tag{15}$$

where

$$K_1^n \equiv \sum_k \{p(t^n,\varphi^n(-\eta_k(t^n,\varphi^n))) - p(t,\varphi(-\eta_k(t^n,\varphi^n)))\} \cdot h_k(t^n,\varphi^n),$$

$$K_2^n \equiv \sum_k p(t,\varphi(-\eta_k(t^n,\varphi^n))) \cdot \left[h_k(t^n,\varphi^n) - h_k(t,\varphi)\right],$$

$$K_3^n \equiv \sum_k \{p(t,\varphi(-\eta_k(t^n,\varphi^n))) - p(t,\varphi(-\eta_k(t,\varphi)))\} \cdot h_k(t,\varphi).$$

Using the Lipschitz property of p one may check that

$$||K_1^n|| \le L_p(|t^n - t| + ||\varphi^n - \varphi||_C) \cdot \sum_k |h_k(t^n,\varphi^n)|. \tag{16}$$

Now we discuss K_2^n. The growth condition of p implies $||p(t,\varphi(-\eta_k(t^n,\varphi^n)))|| \le (C_1||\varphi||_C + C_2)$. Hence

$$||K_2^n|| \le (C_1||\varphi||_C + C_2) \cdot \sum_k |h_k(t^n,\varphi^n) - h_k(t,\varphi)|. \tag{17}$$

In a similar way we obtain

$$||K_3^n|| \le L_p \sum_k |h_k(t,\varphi)| \cdot ||\varphi(-\eta_k(t^n,\varphi^n)) - \varphi(-\eta_k(t,\varphi)||. \tag{18}$$

Now we show that $||K_j^n|| \to 0$ as $n \to \infty$ for $j = 1,2,3$. The first property $||K_1^n|| \to 0$ follows from (A3) and (16). In (17), the series converges uniformly with respect to n since the condition $||\varphi^n - \varphi||_C + |t^n - t| \to 0$ implies that $\{(t,\varphi),(t^n,\varphi^n)\}$ is a bounded subset of $\mathbb{R} \times C$. Assumption (A3) guarantees that each $|h_k(t^n,\varphi^n) - h_k(t,\varphi)|$ is continuous with respect to (t^n,φ^n) and tends to zero when $n \to \infty$. Due to the uniform convergence of the series in (17) (see (A3)), we arrive at $||K_2^n|| \to 0$. To show that $||K_3^n|| \to 0$ we also mention that each $|h_k(t,\varphi)| \cdot ||\varphi(-\eta_k(t^n,\varphi^n)) - \varphi(-\eta_k(t,\varphi)||$ (see (18)) is continuous with respect to (t^n,φ^n) and tends to zero as $n \to \infty$ due to (A3) and the strong continuity of $\varphi \in C$. The uniform convergence (w.r.t. (t^n,φ^n)) of the series in (18) follows from the estimate $|h_k(t,\varphi)| \cdot ||\varphi(-\eta_k(t^n,\varphi^n)) - \varphi(-\eta_k(t,\varphi)|| \le |h_k(t,\varphi)| \cdot 2||\varphi||_C$ (the right-hand side is independent of n!) and the Weierstrass dominant (uniform) convergence theorem. We conclude that $||K_3^n|| \to 0$. Since all $||K_j^n|| \to 0$ as $n \to \infty$ for $j = 1,2,3$ we proved the property $||F_d(t^n,\varphi^n) - F_d(t,\varphi)|| \to 0$. We shown that both F_c and F_d are continuous. The proof of theorem 1 is complete. ∎

In our study we use the standard

Definition 1. *A function $u \in C([a-r,T];X)$ is called a* `mild solution` *on $[a-r,T]$ of the initial value problem (1), (2) if it satisfies (2) and*

$$u(t) = e^{-A(t-a)}\varphi(0) + \int_a^t e^{-A(t-s)} B(s,u_s)\, ds, \quad t \in [a,T]. \tag{19}$$

Theorem 2. *Under the assumptions of theorem 1, the initial value problem (1), (2) possesses a mild solution for any $\varphi \in C$.*

The existence of a mild solution is a consequence of the continuity of $B : \mathbb{R} \times C \to X$, given by theorem 1, which gives us the possibility to use the standard method based on the Schauder fixed point theorem (see [7, theorem 3.1, p.4]).

Theorem 3. *Let all the assumptions of theorem 1 are valid. If additionally $||G(t,u,v)|| \leq k_1(t)(||u|| + ||v||) + k_2(t)$ with k_i are locally integrable on $[a,\infty)$, then a mild solution is global i.e. defined for all $t \geq a$.*

The statement follows from theorem 2 and [34, theorem 2.3, p. 49].

To get the uniqueness of mild solutions we need the following additional assumptions.

(A4) *The total variation of function $g_c \equiv g_{ac} + g_s$ satisfies*

$$\exists L_{Vg_c} \geq 0 : \forall t^1, t^2 \geq a \Rightarrow V^0_{-r}[g_c(\cdot,t^1,\varphi) - g_c(\cdot,t^2,\psi)] \leq L_{Vg_c}(|t^1 - t^2| + ||\varphi - \psi||_C). \tag{20}$$

(A5) *The discrete generating function g_d satisfies the following uniform condition:*

- *there exists continuous function $\eta_{ign}(t) > 0$, such that* `all` *η_k and h_k "*`ignore`*" values of $\varphi(\theta)$ for $\theta \in (-\eta_{ign}(t), 0]$ i.e.*

$$\exists\, \eta_{ign}(t) > 0 : \forall t \geq a, \forall \varphi^1, \varphi^2 \in C : \forall \theta \in [-r, -\eta_{ign}(t)], \Rightarrow \varphi^1(\theta) = \varphi^2(\theta) \quad \Longrightarrow$$

$$\forall k \in \mathbb{N} \Rightarrow \quad \eta_k(t,\varphi^1) = \eta_k(t,\varphi^2) \quad \text{and} \quad h_k(t,\varphi^1) = h_k(t,\varphi^2).$$

Remark. *Assumption (A5) is the natural generalization to the non-autonomous case of multiple discrete state-dependent delays of the condition introduced in [23]. In [23, 25] the function $\eta_{ign}(t)$ was constant $\eta_{ign}(t) \equiv \eta_{ign} > 0$. For more details and examples see [23] and also [25].*

Theorem 4. *Assume (A1)- (A5) are valid, p is as in theorem 1, mapping $G : \mathbb{R} \times X \times X \to X$ is continuous and locally Lipschitz with respect to its second and third coordinates i.e. for any $R > 0$ there exists $L_{G,R} > 0$ such that for all $t > a$, $||u^i||, ||v^i|| \leq R$ one has*

$$||G(t,u^1,v^1) - G(t,u^2,v^2)|| \leq L_{G,R}\left(||u^1 - u^2|| + ||v^1 - v^2||\right). \tag{21}$$

Then initial value problem (1), (2) possesses a `unique` *mild solution on an interval of the form $[a,b)$ where $a < b \leq +\infty$ for any $\varphi \in C$. The solution is continuous with respect to initial data i.e. $||\varphi^n - \varphi||_C \to 0$ implies $||u^n_t - u_t||_C \to 0$ for any $t \in [a,b)$. Here u^n is the unique solution of (1), (2) with initial function φ^n instead of φ.*

Proof of theorem 4. For the simplicity, we first consider a particular case when the generating function $g = g_c \equiv g_{ac} + g_s$ i.e. $F = F_c$ does not contain the discrete delays.

Let $(t^1,\varphi),(t^2,\psi)$ belong to a bounded subset $\mathcal{B}\subset\mathbb{R}\times C$. We use the splitting (11). One can see that (see (13))

$$||I_2||\le M_{\mathcal{B}}\cdot V^0_{-r}[g_c(t^1,\varphi)-g_c(t^2,\psi)]. \qquad (22)$$

Assumption (A4) and (14) imply that F_c is locally Lipschitz i.e. for any $R>0$ there exists $L_{F_c,R}>0$ such that for all $||\varphi||_C\le R$, $||\psi||_C\le R$, $a\le t^1,t^2\le a+R$ one has

$$||F_c(t^1,\phi)-F_c(t^2,\psi)||\le L_{F_c,R}\left(|t^1-t^2|+||\phi-\psi||_C\right). \qquad (23)$$

Consider a sequence $\{\varphi^n\}\subset C$ such that $||\varphi^n-\varphi||_C\to 0$ as $n\to\infty$. Denote by $u(t)=u(t;\varphi)$ *any* mild solution of (1), (2) and by $u^n(t)=u^n(t;\varphi^n)$ *any* mild solution of (1), (2) with initial data $\varphi^n\in C$. The existence of these solutions is proved in theorem 2. The Schauder fixed point theorem (see e.g. [34, theorem 2.1, p.46]), used in the proof of theorem 2 implies that one can choose $R>0$ and $T\in[a,b)$ to have $||\varphi^n||_C\le R$, $||u^n(t;\varphi^n)||\le R$ for all $t\in[a,T]$.

Using the local Lipschitz property of mapping G (21), (23) and the form (3), we get

$$||B(t,\varphi)-B(t,\psi)||\le L_{G,R}(1+L_{F_c,R})||\varphi-\psi||_C+L_{G,R}||F_d(t,\varphi)-F_d(t,\psi)||. \qquad (24)$$

Hence for any $t\in[a,T]$ one has (we remind that $||T(t)||\equiv||e^{-At}||\le e^{\omega t}$ and $F=F_c$)

$$||u_t-u^n_t||_C\le e^{\omega(T-a)}||\varphi-\varphi^n||_C+L_{G,R}(1+L_{F_c,R})e^{\omega(T-a)}\cdot\int_a^t||u_s-u^n_s||_C\,ds.$$

The last estimate (by the Gronwall lemma) implies

$$||u_t-u^n_t||_C\le e^{L_{G,R}(1+L_{F_c,R})e^{\omega(T-a)}}e^{\omega(T-a)}\cdot||\varphi-\varphi^n||_C.$$

That is

$$||u_t-u^n_t||_C\le C_T\cdot||\varphi-\varphi^n||_C,\quad \forall t\in[a,T],\ C_T\equiv e^{\omega(T-a)}\exp\{L_{G,R}(1+L_{F_c,R})e^{\omega(T-a)}\}. \qquad (25)$$

It proves the uniqueness of mild solutions and the continuity with respect to initial data in the case $g=g_c$.

The second particular case $g=g_d$ (the purely discrete delay) and only one point of discontinuity has been considered in detail in [23] (the autonomous case). It was proved in [23] that (A5) implies the desired result.

Now we consider the general case (both discrete and distributed delays, including the case of multiple discrete SD-delays).

Using the splitting $F=F_d+F_c$, we have, by definition of mild solutions,

$$u^n(t)-u(t)=e^{-A(t-a)}(\varphi^n(0)-\varphi(0))+\int_a^t e^{-A(t-\tau)}\left\{F_d(\tau,u^n_\tau)-F_d(\tau,u_\tau)\right\}\,d\tau$$

$$+\int_a^t e^{-A(t-\tau)}\left\{F_c(\tau,u^n_\tau)-F_c(\tau,u_\tau)\right\}\,d\tau.$$

Using (23), one gets for all $t\in[a,T]$

$$||u^n(t)-u(t)||\le||\varphi^n(0)-\varphi(0)||e^{\omega(T-a)}+e^{\omega(T-a)}L_{G,R}\int_a^t||F_d(\tau,u^n_\tau)-F_d(\tau,u_\tau)||\,d\tau$$

$$+L_{G,R}(1+L_{F_c,R})e^{\omega(T-a)}\int_a^t ||u_\tau^n - u_\tau||_C\,d\tau$$

$$= G^n(t) + L_{G,R}(1+L_{F_c,R})e^{\omega(T-a)}\int_a^t ||u_\tau^n - u_\tau||_C\,d\tau, \tag{26}$$

where

$$G^n(t) \equiv ||\varphi^n(0)-\varphi(0)||e^{\omega(T-a)} + L_{G,R}e^{\omega(T-a)}\int_a^t ||F_d(\tau,u_\tau^n) - F_d(\tau,u_\tau)||\,d\tau \tag{27}$$

is a nondecreasing (in time) function. Multiply the last estimate by $e^{-tL_{G,R}(1+L_{F_c,R})e^{\omega(T-a)}}$ to get

$$\frac{d}{dt}\left(e^{-tL_{G,R}(1+L_{F_c,R})e^{\omega(T-a)}}\int_a^t ||u_\tau^n - u_\tau||_C\,d\tau\right) \le e^{-tL_{G,R}(1+L_{F_c,R})e^{\omega(T-a)}}G^n(t),$$

which, after integration from a to t, shows that ($G^n(t)$ is nondecreasing)

$$e^{-tL_{G,R}(1+L_{F_c,R})e^{\omega(T-a)}}\int_a^t ||u_\tau^n - u_\tau||_C\,d\tau \le \int_a^t e^{-\tau L_{G,R}(1+L_{F_c,R})e^{\omega(T-a)}}G^n(\tau)\,d\tau$$

$$\le G^n(t)\int_a^t e^{-\tau L_{G,R}(1+L_{F_c,R})e^{\omega(T-a)}}\,d\tau$$

$$= G^n(t)\left(e^{-aL_{G,R}(1+L_{F_c,R})e^{\omega(T-a)}} - e^{-tL_{G,R}(1+L_{F_c,R})e^{\omega(T-a)}}\right)[L_{G,R}(1+L_{F_c,R})]^{-1}e^{-\omega(T-a)}.$$

We have

$$L_{G,R}(1+L_{F_c,R})e^{\omega(T-a)}\int_a^t ||u_\tau^n - u_\tau||_C\,d\tau \le G^n(t)\left(e^{(t-a)L_{G,R}(1+L_{F_c,R})e^{\omega(T-a)}} - 1\right).$$

We substitute the last estimate into (26) to obtain

$$||u_t^n - u_t||_C \le G^n(t)\cdot e^{(t-a)L_{G,R}(1+L_{F_c,R})e^{\omega(T-a)}}. \tag{28}$$

Let us fix any $c > b$ and denote by $\eta_{ign} \equiv \min\{\eta_{ign}(t) : t \in [a,c]\}$. By Assumption (A5), $\eta_{ign}(t) > 0$ and continuous, so $\eta_{ign} > 0$. Let us denote by $\sigma \equiv \min\{b, a+\eta_{ign}\} > a$.

Now our goal is to show that for any fixed $t \in [a,\sigma)$ one has $G^n(t) \to 0$ when $n \to \infty$ (we remind that $||\varphi^n - \varphi||_C \to 0$).

Let us consider the extension functions

$$\overline{\varphi}(s) \equiv \left[\begin{array}{ll} \varphi(s) & s \in [-r,0]; \\ \varphi(0) & s \in (0,\sigma) \end{array}\right. \quad \text{and} \quad \overline{\varphi}^n(s) \equiv \left[\begin{array}{ll} \varphi^n(s) & s \in [-r,0]; \\ \varphi^n(0) & s \in (0,\sigma) \end{array}\right. .$$

An important consequence of (A5) is that $F_d(t,u_t) = F_d(t,\bar\varphi_t)$ for all $t \in [a,\sigma)$ and any solution $u : [a-r,\sigma) \to X$, satisfying $u_a = \varphi$. In the same way $F_d(t,u_t^n) = F_d(t,\bar\varphi_t^n)$ for all $t \in [a,\sigma)$. Hence the continuity of F_d implies $||F_d(\tau,\overline{\varphi}_\tau^n) - F_d(\tau,\overline{\varphi}_\tau)|| \to 0$ for any $\tau \in [a,\sigma)$.

Remark. *We notice that the case we consider now is simpler that the one in the proof of theorem 1 (see (15)) since we estimate F_d at the same first coordinate (time moment $\tau \in [a,\sigma)$).*

The property $||F_d(\tau,\overline{\varphi}^n_\tau) - F_d(\tau,\overline{\varphi}_\tau)|| \to 0$ for any $\tau \in [a,\sigma)$ and the uniform boundedness of the term allows us to use the classical Lebesgue-Fatou lemma (see [35, p.32]) for the scalar function $||F_d(\tau,\overline{\varphi}^n_\tau) - F_d(\tau,\overline{\varphi}_\tau)||$ to conclude that $G^n(t) \to 0$ when $n \to \infty$ (for any fixed $t \in [a,\sigma)$). Hence (28) gives the continuity of the mild solutions with respect to initial functions for all $t \in [a,\sigma)$. Particularly, it gives the uniqueness of solutions. For bigger time values we use the chain rule (by the uniqueness) for steps less than or equal to, say $(\sigma - a)/2$ (for more details see [23]). Since the composition of continuous mappings is continuous, the proof of theorem 4 is complete. ∎

Remark. *Discussing the proof of theorem 4, we see that in the case $g = g_c$ (no discrete state-dependent delays) the delay mapping $F = F_c$ is locally Lipschitz continuous (see (23)) and, as a consequence, one has a standard estimate for the difference of two solutions (25) in terms of the difference of initial functions $||\varphi - \varphi^n||_C$. In case of the presence of discrete SDDs we have estimate (28) with G^n defined by (27) and for the Lebesgue-Fatou lemma it was enough to have property $||F_d(\tau,\overline{\varphi}^n_\tau) - F_d(\tau,\overline{\varphi}_\tau)|| \to 0$ which does not provide information on the difference $||F(t,\varphi^n) - F(t,\varphi)||$ in terms of $||\varphi-\varphi^n||_C$ (it is definitely not a Lipschitz property). A way to get such an information is to use the modulus of continuity $\omega_f(\delta;Y)$. We remind that $\omega_f(\delta;Y) \equiv \sup\{||f(x)-f(y)|| : x,y \in Y, ||x-y|| \le \delta\}$. For the simplicity of presentation we consider F_d with one discrete SDD. We have (see (15) with $t = t^n = \tau$)*

$$F_d(\tau,\varphi^n) - F_d(\tau,\varphi) = p(\tau,\varphi^n(-\eta(\tau,\varphi^n))) \cdot h(\tau,\varphi^n) - p(\tau,\varphi(-\eta(\tau,\varphi))) \cdot h(\tau,\varphi). \quad (29)$$

The estimates for $K^n_i, i = 1,2,3$ (see (16)- (18) with $k = 1$) show that

$$||F_d(\tau,\varphi^n) - F_d(\tau,\varphi)|| \le L_p V^0_{-r} g_d \cdot ||\varphi - \varphi^n||_C$$

$$+(C_1||\varphi||_C + C_2) \cdot \omega_h\left(||\varphi-\varphi^n||_C;\tilde{Y}\right) + L_p V^0_{-r} g_d \cdot \omega_\varphi\left(\omega_\eta\left(||\varphi-\varphi^n||_C;\tilde{Y}\right);[-r,0]\right),$$

where we denoted by $\tilde{Y} \equiv \{(t,\bar{\varphi}_t),(t,\bar{\varphi}^n_t) : t \in [a,\sigma_1], n \in \mathbb{N}\}$, with $a < \sigma_1 < \sigma$. By (A1) one has $V^0_{-r} g_d \le M_{Vg}$ and

$$||F_d(\tau,\varphi^n) - F_d(\tau,\varphi)|| \le L_p M_{Vg} \cdot \left[||\varphi-\varphi^n||_C + \omega_\varphi\left(\omega_\eta\left(||\varphi-\varphi^n||_C;\tilde{Y}\right);[-r,0]\right)\right]$$

$$+(C_1||\varphi||_C + C_2) \cdot \omega_h\left(||\varphi-\varphi^n||_C;\tilde{Y}\right). \quad (30)$$

Since $\varphi^n \to \varphi$ in C, we see that $\tilde{Y}$ is compact. Using (A3) and the classical Cantor theorem, we know that h and η are equicontinuous on $\tilde{Y}$ and $\omega_h\left(||\varphi-\varphi^n||_C;\tilde{Y}\right) \to 0$ and $\omega_\eta\left(||\varphi-\varphi^n||_C;\tilde{Y}\right) \to 0$ as $||\varphi-\varphi^n||_C \to 0$. We remind that $F_d(t,u_t) = F_d(t,\bar{\varphi}_t)$, $F_d(t,u^n_t) = F_d(t,\bar{\varphi}^n_t)$ and use $||\bar{\varphi}_t - \bar{\varphi}^n_t||_C \le ||\varphi-\varphi^n||_C$. Finally, one can substitute the estimate (3) into (27) and then the estimate for G^n into (28) to get an estimate for the difference of two solutions in terms of the difference of initial functions $||\varphi-\varphi^n||_C$.

Remark. *In the theory of* `ordinary` *differential equations with SDDs it is usual to restrict the class of initial functions φ to Lipschitz ones [31, 12]. In this case one restricts the set of SDDs (both η and h) to Lipschitz mappings. In such a situation the previous remark evidently provides the Lipschitz property of F_d and hence F. It follows from the property $f \in \mathcal{L}ip(L_f;Y) \Longrightarrow \omega_f(\delta;Y) \le L_f \cdot \delta$ and the estimates above.*

4 Invariance

This section is devoted to an extension of the fundamental invariance principle [18] to the case of PDEs with discrete state-dependent delays.

We also refer the reader to [26] for important generalizations of the invariance principle in several directions (without SDDs) and for a survey of the existing literature on the subject. In the present paper the emphasis is on the delayed term, not on the partial differential operator.

Following [18], we assume that the next hypotheses are satisfied:

(H1) D is a closed subset of $[a-r,\infty)\times X$ and $D(t)\equiv\{x\in X:(t,x)\in D\}$ is nonempty for each $t\geq a-r$.

(H2) $\mathcal{D}$ is the closed subset of $[a,\infty)\times C$ defined by $\mathcal{D}\equiv\{(t,\varphi):\varphi(\theta)\in D(t+\theta)$ for all $-r\leq\theta\leq 0\}$. Also, $\mathcal{D}(t)\equiv\{\varphi\in C:(t,\varphi)\in\mathcal{D}\}$ for each $t\geq a$, and we assume that $\mathcal{D}(t)$ is nonempty for each set $t\geq a$.

(H3) For each $b>a$ there are a $\hat{K}(b)>0$ and a continuous nondecreasing function $\eta_b:[0,b-a)\to[0,\infty)$ satisfying $\eta_b(0)=0$ with the property that if $a\leq t_1<t_2\leq b, x_1\in D(t_1)$, and $x_2\in D(t_2)$, then there is a continuous function $w:[t_1,t_2]\to X$ such that $w(t_1)=x_1, w(t_2)=x_2, w(t)\in D(t)$ for $t_1<t<t_2$, and

$$|w(t)-w(s)|\leq\eta_b(|t-s|)+\hat{K}(b)|t-s|\frac{|x_2-x_1|}{t_2-t_1}$$

for all $s,t\in[t_1,t_2]$.

(H4) B is continuous from $D(B)$ into X where $\mathcal{D}\subset D(B)\subset[a,\infty)\times C$.

Remark [18, page 16]. *If D is* `convex` *then (H3) is automatically satisfied by defining*

$$w(t)=\frac{(t_2-t)x_1+(t-t_1)x_2}{(t_2-t_1)}\qquad \textit{for}\qquad t_1\leq t\leq t_2.$$

We see that $||w(t)-w(s)||=\left\|\frac{(s-t)x_1+(t-s)x_2}{(t_2-t_1)}\right\|\leq\frac{||x_1-x_2||}{(t_2-t_1)}|t-s|$ *for* $t_1\leq s<t\leq t_2$ *and hence (H3) is satisfied with* $\hat{K}(b)\equiv 1$ *and* $\eta_b=0$.

We use the notation

$$d(x;D(t))\equiv\inf\{|x-y|:y\in D(t)\}\quad\text{for}\qquad x\in X,t\geq a.$$

The fundamental criterion for the invariance of the set $\mathcal{D}$, called the *subtangential condition*, (see [18, (2.2)]) is given by

$$\lim_{h\to0+}\frac{1}{h}\,d\left(e^{-Ah}\varphi(0)+\int_t^{t+h}e^{-A(t+h-s)}B(t,\varphi)\,ds\,;D(t+h)\right)\quad\text{for}\qquad(t,\varphi)\in\mathcal{D}.\quad(31)$$

The following result is an extension of [18, theorem 2] to the case of general state-dependent delay.

Theorem 5. *Assume (A1)- (A5), (H1)-(H4) and (31) are valid. Let mapping p be as in theorem 1 and* $G : \mathbb{R} \times X \times X \to X$ *satisfy the property: for each* $R > 0$ *there are an* $L_{G,R} > 0$ *and a continuous* $\nu_R : [0, \infty) \to [0, \infty)$ *such that* $\nu_R(0) = 0$ *and*

$$||G(t^1, u^1, v^1) - G(t^2, u^2, v^2)|| \le \nu_R(|t^1 - t^2|) + L_{G,R} \left(||u^1 - u^2|| + ||v^1 - v^2||\right) \qquad (32)$$

for all $||u^i||, ||v^i|| \le R$ *and* $a \le t^1, t^2 \le a + R$.

Then initial value problem (1), (2) has a unique mild solution on an interval of the form $[a, b)$ *where* $a < b \le +\infty$ *for any* $\varphi \in C$. *If additionally* $\varphi \in \mathcal{D}(a)$, *then* $u(t) \in D(t)$ *for* $a \le t < b$ *and if* $b < +\infty$ *then* $||u_t||_C \to \infty$ *as* $t \to b - 0$.

Remark. *In contrast to [18, theorem 2], the nonlinear term B in equation (1) is not Lipschitz (with respect to the second coordinate, c.f. [18, property (2.3), p.18]) in the presence of discrete SDD. So [18, theorem 2] could not be applied to our case. Instead, assumption (A5) provides the uniqueness of mild solutions and saves the line of the proof presented in [18].*

The proof of theorem 5 follows closely that of [18, theorem 2]. The last consists of eight lemmas and the final part which spends pages 35-43 of the original article. There is no need to repeat these lemmas since they are not affected by the lack of the Lipschitz property of B and we refer the reader to [18] for all notations and details. Here we only remind the main steps of the original proof and give the new part of the proof based on assumption (A5).

First, for fixed $\sigma > a, \varepsilon_0 > 0$ and any $\varepsilon \in [0, \varepsilon_0]$ the ε-approximate solution w is constructed. It is done by a careful construction (see [18] for all details) of an increasing sequence $\{t_i\}_0^\infty \subset [a, \sigma + \varepsilon_0]$ such that $w(t_i) \in D(t_i)$ and (see [18, (4.6)])

$$\left\| e^{-A(t_{i+1}-t_i)} w(t_{t_i}) + \int_{t_i}^{t_{i+1}} e^{-A(t_{i+1}-s)} B(t_i, w_{t_i})\, ds \; - w(w_{t_{i+1}}) \right\| \quad \le \varepsilon(t_{i+1} - t_i). \qquad (33)$$

Let $\{\varepsilon_n\}_1^\infty$ be a decreasing sequence such that $\varepsilon_n \to 0$ as $n \to \infty$ and for each $n \ge 1$ let w^n and $\{t_i^n\}_{i=0}^\infty$ be as constructed above with $\varepsilon = \varepsilon_n, t_i = t_i^n$, and $w = w^n$. Denote by $\gamma^n : [a, \sigma] \to [a, \sigma]$ the function $\gamma^n(t) = t_i^n$ whenever $t \in [t_i^n, t_{i+1}^n]$.

Remark A. *In addition to consideration in [18], we assume that* $\sigma + \varepsilon_0 - a < \eta_{ign} \equiv \min_{t\in[a,c]} \eta(t)$ *for some fixed* $c > a$. *Since we prove the local existence, c could be chosen arbitrary and (A5) gives* $\eta_{ign} > 0$ *(*$\eta(t)$ *is continuous).*

For convenience a companion function v^n for w^n is defined in the following manner (see [18, (4.9)])

$$v^n(t) = e^{-A(t-a)}\varphi(0) + \int_a^t e^{-A(t-s)} B(\gamma^n(s), w^n_{\gamma^n(s)})\, ds \quad \text{for } t \in [a, \sigma]. \qquad (34)$$

and $v^n(a + \theta) = \varphi(\theta)$ for $\theta \in [-r, 0]$.

It is shown (see [18, lemmas 4.6 and 4.7)]) that

$$||v^n(t) - w^n(t)|| \le \hat{P} \max\{\varepsilon_n, \eta_b(\varepsilon_n)\} \quad \text{for} \quad t \in [a - r, \sigma],\, n = 1, 2, \ldots,\, \hat{P} > 0, \qquad (35)$$

$$||w^n_t - w^n_{\gamma^n(t)}||_C \le \hat{Q} \max\{\varepsilon_n, \eta_b(\varepsilon_n)\} \quad \text{for} \quad t \in [a, \sigma],\, n = 1, 2, \ldots,\, \hat{Q} > 0, \qquad (36)$$

with $\hat{P}, \hat{Q} > 0$ both independent of t and n.

Next, [18, lemma 4.8] shows that *if* there is a function $u : [a-r,\sigma] \to X$ such that $u(t) = \lim_{n\to\infty} w^n(t)$ uniformly for $t \in [a-r,\sigma]$, *then* $(t, u(t)) \in D$ and u is a *solution* to (1), (2) on $[a,\sigma]$.

Now we proceed the final part of the proof where we use assumption (A5) instead of the Lipschitz property of B. First, we need an estimate for $||B(t^1, u_{t^1}) - B(t^2, u_{t^2})||$. We consider the splitting of the delay mapping F onto continuous and discrete parts $F = F_c + F_d$ (according to (10)) and use the local Lipschitz property of F_c (due to (A4), see (23)). More precisely, using the local Lipschitz property of mapping G (32), (23) and the form (3), we get

$$||B(t^1, u_{t^1}) - B(t^2, u_{t^2})||$$
$$\le \nu_R\left(|t^1 - t^2|\right) + L_{G,R}L_{F_c,R} \cdot |t^1 - t^2| + L_{G,R}(1 + L_{F_c,R}) \cdot ||u_{t^1} - u_{t^2}||_C$$
$$+ L_{G,R}\,||F_d(t^1, u_{t^1}) - F_d(t^2, u_{t^2})||. \tag{37}$$

Since $|t - \gamma^n(t)| \le \varepsilon_n$ it follows that

$$|\gamma^n(t) - \gamma^m(t)| \to 0 \quad \text{as} \quad n,m \to \infty \quad \text{uniformly for} \quad t \in [a,\sigma]. \tag{38}$$

Using (35), (36), consider $\bar{R} > 0$ such that $||w^n(t)|| \le \bar{R}$ for all $n \ge 1$ and $t \in [a-r,\sigma]$. Then (37) implies

$$||B(\gamma^n(s), w^n_{\gamma^n(s)}) - B(\gamma^m(s), w^m_{\gamma^m(s)})||$$
$$\le \nu_R\left(|\gamma^n(s) - \gamma^m(s)|\right) + L_{G,R}L_{F_c,R} \cdot |\gamma^n(s) - \gamma^m(s)| + L_{G,\bar{R}}(1 + L_{F_c,R})||w^n_{\gamma^n(s)} - w^m_{\gamma^m(s)}||_C$$
$$+ L_{G,\bar{R}}||F_d(\gamma^n(s), w^n_{\gamma^n(s)}) - F_d(\gamma^m(s), w^m_{\gamma^m(s)})|| \le [\text{estimates (35), (36) give}]$$
$$\le L_{G,\bar{R}}(1 + L_{F_c,R})||v^n_s - v^m_s||_C + L_{G,\bar{R}}||F_d(\gamma^n(s), w^n_{\gamma^n(s)}) - F_d(\gamma^m(s), w^m_{\gamma^m(s)})|| + \widetilde{\varepsilon}_{n,m},$$

where $\widetilde{\varepsilon}_{n,m} \to 0$ as $n,m \to \infty$ (due to (38)). Using (34), we obtain

$$||v^n(t) - v^m(t)|| \le \int_a^t ML_{G,\bar{R}}(1 + L_{F_c,R})||v^n_s - v^m_s||_C\, ds$$
$$+ \int_a^t ML_{G,\bar{R}}||F_d(\gamma^n(s), w^n_{\gamma^n(s)}) - F_d(\gamma^m(s), w^m_{\gamma^m(s)})||\, ds + \hat{\varepsilon}_{n,m}, \tag{39}$$

where $\hat{\varepsilon}_{n,m} \to 0$ as $n,m \to \infty$.

Now our goal is to show that the second integral in (39) tends to zero as $n,m \to \infty$. Let us denote by $\bar{\varphi}$ the extension function $\bar{\varphi}(a+\theta) = \varphi(\theta)$ for $\theta \in [-r,0]$ and $\bar{\varphi}(s) = \varphi(0)$ for $s \in (a,\sigma]$. We remind that $\sigma - a < \eta_{ign}$ (see remark A above). An important consequence of (A5) is that $F_d(t,u_t) = F_d(t,\bar{\varphi}_t)$ for all $t \in [a,\sigma]$ and any continuous function $u : [a-r,\sigma] \to X$, satisfying $u_a = \varphi$. Since all w^n, by construction, are continuous and satisfy $w^n_a = \varphi$, we arrive to the property $F_d(t, w^n_t) = F_d(t,\bar{\varphi}_t)$ for all $t \in [a,\sigma]$. Hence (see the second integral in (39))

$$G^{n,m}_d(s) \equiv ||F_d(\gamma^n(s), w^n_{\gamma^n(s)}) - F_d(\gamma^m(s), w^m_{\gamma^m(s)})|| = ||F_d(\gamma^n(s), \bar{\varphi}_{\gamma^n(s)}) - F_d(\gamma^m(s), \bar{\varphi}_{\gamma^m(s)})||.$$

The continuity of F_d and (38) give $G^{n,m}_d(s) \to 0$ as $n,m \to \infty$ for all $s \in [a,\sigma]$. Since $G^{n,m}_d(s)$ is bounded, the classical Lebesgue-Fatou lemma implies that

$$\int_a^t G^{n,m}_d(s)\, ds \to 0 \text{ as } n,m \to \infty.$$

The last property gives (see (39))

$$||v^n(t) - v^m(t)|| \leq \int_a^t ML_{G,\bar{R}}(1 + L_{F_c,R})||v_s^n - v_s^m||_C \, ds + \varepsilon_{n,m}, \tag{40}$$

where $\varepsilon_{n,m} \to 0$ as $n, m \to \infty$.

The rest of the proof follows [18, page 43]. Defining $q_{n,m}(t) \equiv \max\{||v^n(s) - v^m(s)|| : a - r \leq s \leq t\}$ we see that for each $t \in [a, \sigma]$ there is an $\alpha(t) \in [a - r, t]$ such that

$$q_{n,m}(t) = ||v^n(\alpha(t)) - v^m(\alpha(t))|| \leq \int_a^{\alpha(t)} ML_{G,\bar{R}}(1 + L_{F_c,R})||v_s^n - v_s^m||_C \, ds + \varepsilon_{n,m}$$

$$\leq \int_a^{\alpha(t)} ML_{G,\bar{R}}(1 + L_{F_c,R}) \, q_{n,m}(s) \, ds + \varepsilon_{n,m}.$$

Gronwall's inequality along with the fact that $\varepsilon_{n,m} \to 0$ as $n, m \to \infty$ shows that $q_{n,m}(t) \to 0$ as $n, m \to \infty$, and hence $\{v^n(t)\}_{n=1}^{\infty}$ is uniformly Cauchy on $[a-r, \sigma]$. This implies that $\{w^n(t)\}_{n=1}^{\infty}$ is uniformly Cauchy on $[a-r, \sigma]$ and hence initial value problem (1), (2) has a mild solution on $[a-r, \sigma]$ (see the discussion above and [18, lemma 4.8]). The uniqueness of solution is due to (A5) and provided by theorem 3. The standard continuation arguments give solutions defined on a maximal interval. The proof of theorem 5 is complete. ∎

The following important corollary remains valid in the presence of SDDs.

Corollary (c.f. [18, page 18]). *Suppose K is a closed, convex subset of X and all the assumptions of theorem 5 are satisfied with $D(t) \equiv K$ for all $t \geq a$. Suppose further that*

(a) $T(t) : K \to K$ *for* $t \geq 0$ *and*

(b) $\lim_{h \to 0+} \frac{1}{h} d(\varphi(0) + hB(t, \varphi); K) = 0$ *for* $(t, \varphi) \in \mathcal{D}$.

Then (1), (2) has a unique noncontinuable mild solution u on $[a, b)$ for some $b > a$ and $u(t) \in K$ for all $t \in [a - r, b)$.

Since we are interested in models from biology (see the examples above) the following remark is of prime importance for us.

Remark (c.f. [18, page 7]). *Consider system (8). If $K = [0, \infty)^m$, then condition (a) of the previous corollary holds and condition (b) holds only in case $G = (G^i)_1^m$ is quasipositive: if $k \in \{1, \ldots, m\}$ and $(t, \varphi) \in [a, \infty) \times C([-r, 0]; C(\bar{\Omega})^m)$ with $\varphi^i(\theta, x) \geq 0$ for all $-r \leq \theta \leq 0, x \in \bar{\Omega}$ and $i = 1, \ldots, m$, then $\varphi^i(0, \cdot) = 0$ implies $G^i(t, F^i(t, \varphi(\cdot, x))) \geq 0$ for all $x \in \bar{\Omega}$. This condition gives criteria to determine if solutions to (8) remain nonnegative if they are nonnegative initially.*

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